Algorithms and complexity for metric dimension and location-domination on interval and permutation graphs

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joint work with:

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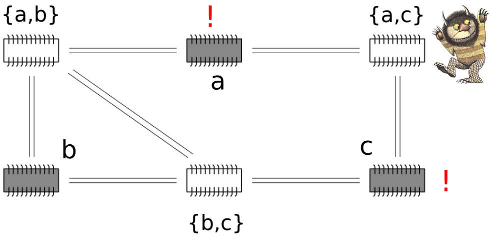
Definition - Locating-dominating set (Slater, 1980's)

$D \subseteq V(G)$ locating-dominating set of G:

- for every $u \in V$, $N[v] \cap D \neq \emptyset$ (domination).
- $\forall u \neq v$ of $V(G) \setminus D$, $N(u) \cap D \neq N(v) \cap D$ (location).

Motivation: fault-detection in networks.
$\rightarrow$ The set D of grey processors is a set of fault-detectors.



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Notation. location-domination number $LD(G)$: smallest size of a locating-dominating set of G

Domination number: $\gamma(P_n) = \lceil \frac{n}{3} \rceil$

Location-domination number: $LD(P_n) = \lceil \frac{2n}{5} \rceil$

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Notion related to test covers in hypergraphs (also known as separating systems, distinguishing transversals...)

Theorem (Slater, 1980's)

G graph of order n, $LD(G) = k$.
Then $n \leq 2^k + k - 1$, i.e. $LD(G) = \Omega(\log n)$.

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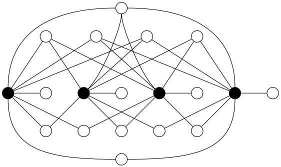
Tight example ($k = 4$): 

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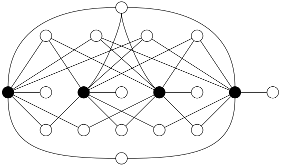
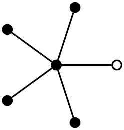
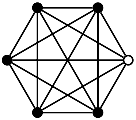


Graphs G with large $LD(G)$:





Complexity of LOCATING-DOMINATING SET

LOCATING-DOMINATING SET

INPUT: Graph G, integer k.
QUESTION: Is there a locating-dominating set of G of size k?

- polynomial for:
  - graphs of bounded cliquewidth via MSOL (Courcelle's theorem)
  - chain graphs (Fernau, Heggernes, van't Hof, Meister, Saei, 2015)

- NP-complete for:
  - bipartite graphs (Charon, Hudry, Lobstein, 2003)
  - planar bipartite unit disk graphs (Müller & Sereni, 2009)
  - planar graphs, arbitrary girth (Auger, 2010)
  - planar bipartite subcubic graphs (F. 2013)
  - co-bipartite graphs, split graphs (F. 2013)
  - line graphs (F., Gravier, Naserasr, Parreau, Valicov, 2013)

LOCATING-DOMINATING SET

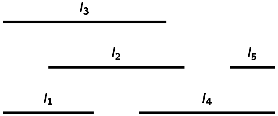
INPUT: Graph G, integer k.
QUESTION: Is there a locating-dominating set of G of size k?

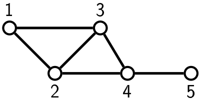
- Trivially FPT for parameter k because $n \leq 2^k + k - 1$: whole graph is kernel.

$\longrightarrow n^{O(k)} = 2^{k^{O(k)}}$-time brute-force algorithm

Definition - Interval graph

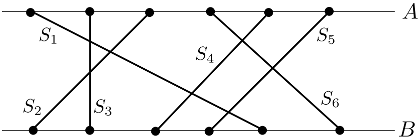
Intersection graph of intervals of the real line.

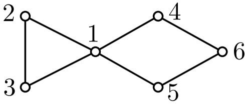




Definition - Permutation graph

Given two parallel lines A and B:
intersection graph of segments joining A and B.





Complexity - Interval and permutation graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

LOCATING-DOMINATING SET is NP-complete for graphs that are both interval and permutation.

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Reduction from 3-DIMENSIONAL MATCHING:

- INPUT: A, B, C sets and $\mathscr{T} \subset A \times B \times C$ triples
- QUESTION: is there a perfect 3-dimensional matching $M \subset T$, i.e., each element of $A \cup B \cup C$ appears exactly once in M?

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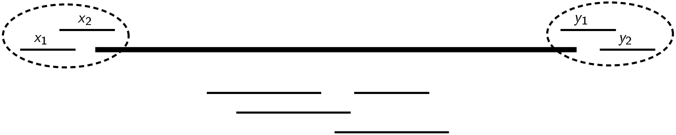
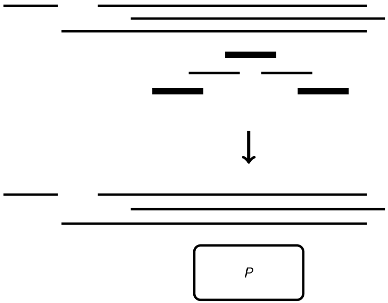
Main idea: an interval can separate pairs of intervals **far away** from each other (without affecting what lies in between)

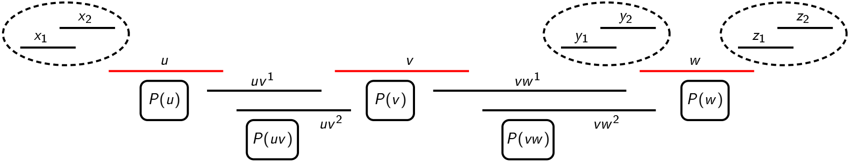


Dominating gadget: ensure all intervals are dominated and most, separated.



Transmitter gadget: to separate $\{uv^1, uv^2\}$ and $\{vw^1, vw^2\}$, either:

1. take only v into solution, or
2. take both u, w — and separate pairs $\{x_1, x_2\}$, $\{y_1, y_2\}$, $\{z_1, z_2\}$ "for free".



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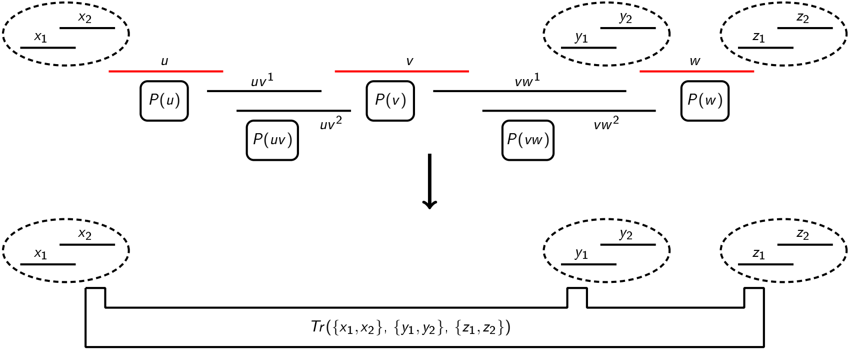
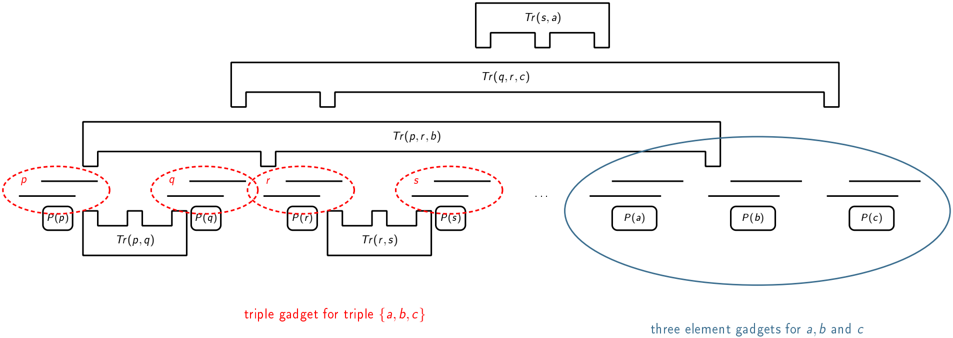
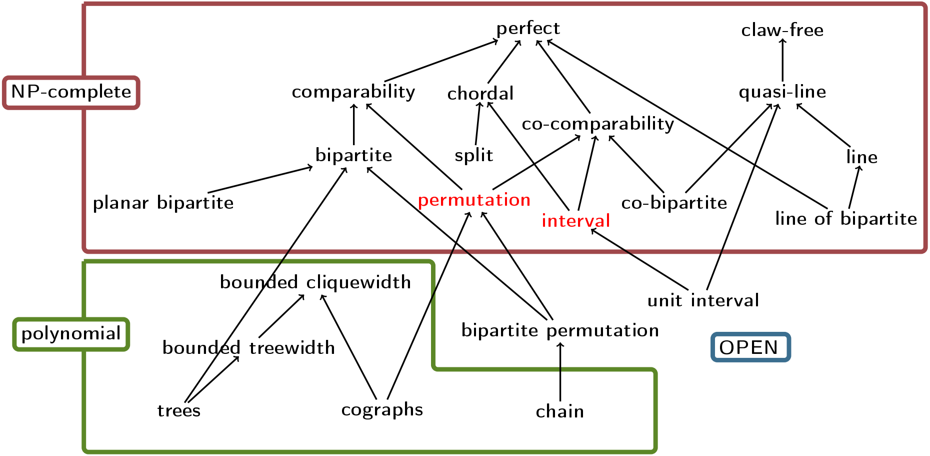


3DM instance on $3n$ elements, m triples.

$$\exists \text{ 3-dimensional matching} \iff LD(G) \leq 94m + 10n$$





NP-complete
perfect
claw-free
comparability
chordal
quasi-line
co-comparability
bipartite
split
line
planar bipartite
permutation
interval
co-bipartite
line of bipartite
polynomial
bounded cliquewidth
unit interval
bounded treewidth
bipartite permutation
OPEN
trees
cographs
chain

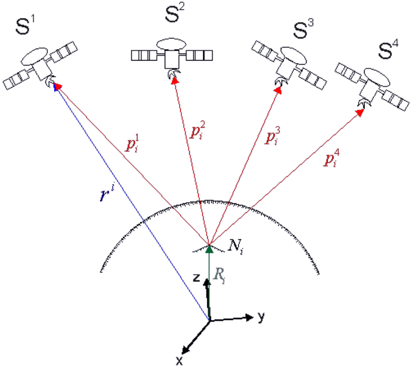
Now, $w \in V(G)$ separates $\{u, v\}$ if $dist(w, u) \neq dist(w, v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)

$R \subseteq V(G)$ resolving set of G:

$$\forall u \neq v \text{ in } V(G), \text{ there exists } w \in R \text{ that separates } \{u, v\}.$$

Motivation ("GPS" system): position determined by distances to 4 satellites



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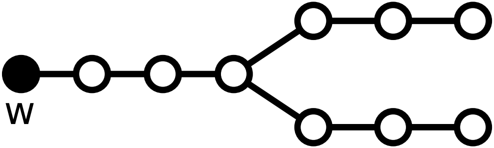

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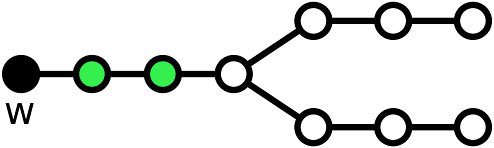

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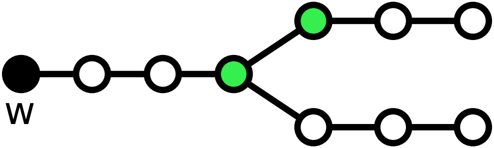

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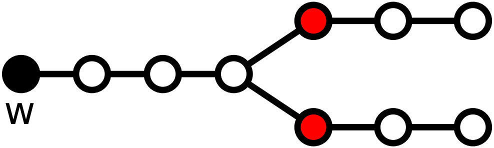

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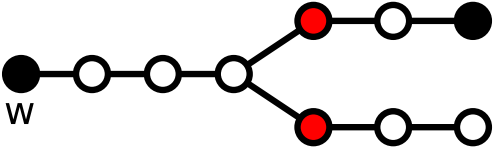

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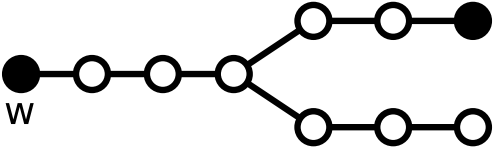

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$MD(G)$: metric dimension of G, minimum size of a resolving set of G.

Remark: $MD(G) \leq LD(G)$.

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$$MD(G) = 1 \iff G \text{ is a path}$$

Complexity of METRIC DIMENSION

METRIC DIMENSION

INPUT: Graph G, integer k.
QUESTION: Is there a resolving set of G of size k?

- polynomial for:
  - trees (simple leg rule: Slater, 1975)
  - outerplanar graphs (Díaz, van Leeuwen, Pottonen, Serna, 2012)
  - bounded cyclomatic number (Epstein, Levin, Woeginger, 2012)
  - cographs (Epstein, Levin, Woeginger, 2012)
  - chain graphs (Fernau, Heggernes, van't Hof, Meister, Saei, 2015)

- NP-complete for:
  - general graphs (Garey & Johnson, 1979)
  - planar graphs (Díaz, van Leeuwen, Pottonen, Serna, 2012)
  - bipartite, co-bipartite, line, split graphs (Epstein, Levin, Woeginger, 2012)
  - Gabriel unit disk graphs (Hoffmann & Wanke, 2012)

METRIC DIMENSION

INPUT: Graph G, integer k.
QUESTION: Is there a resolving set of G of size k?

- W[2]-hard for parameter k, even for bipartite subcubic graphs (Hartung & Nichterlein, 2013)

- Trivially FPT when diameter $D = f(k)$ since $n \leq D^k + k$:

$\rightarrow$ whole graph is kernel (example: split graphs, co-bipartite graphs)

G graph of diameter 2. S resolving set of G.

$\rightarrow$ Every vertex in $V(G) \setminus S$ is distiguished by its neighborhood within S

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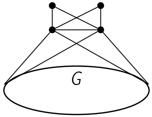
Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

LOCATING-DOMINATING SET is NP-complete for graphs that are both interval and permutation.

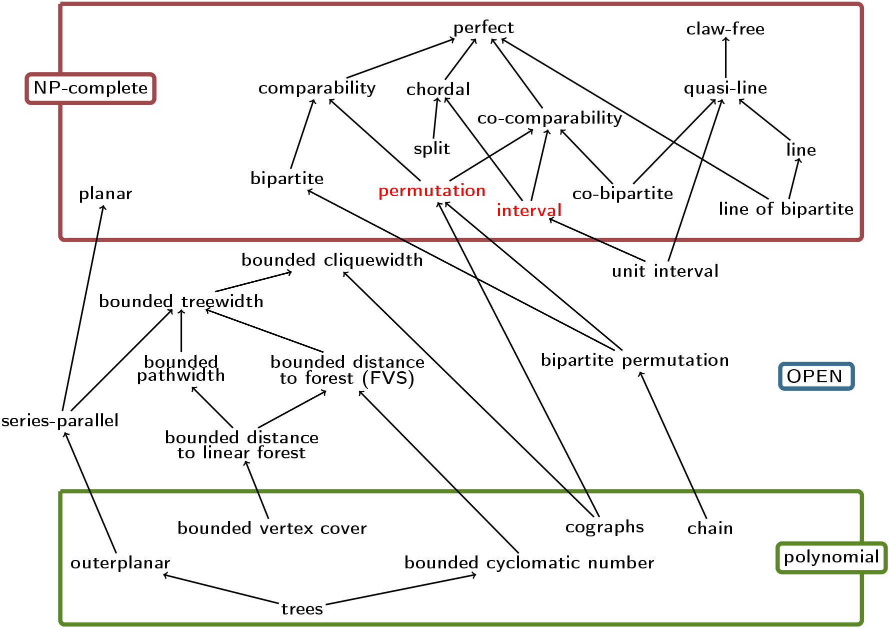
Reduction from LOCATING-DOMINATING SET to METRIC DIMENSION:



$MD(G) = LD(G) + 2$

Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

METRIC DIMENSION is NP-complete for graphs that are both interval and permutation (and have diameter 2).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permutation
interval
co-bipartite
line of bipartite
planar
bounded cliquewidth
unit interval
bounded treewidth
bounded pathwidth
bounded distance to forest (FVS)
bipartite permutation
OPEN
series-parallel
bounded distance to linear forest
bounded vertex cover
cographs
chain
polynomial
outerplanar
bounded cyclomatic number
trees

Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

METRIC DIMENSION can be solved in time $2^{O(k^4)}n$ on interval graphs.

(Recall: METRIC DIMENSION W[2]-hard for parameter k)

FPT algorithm for METRIC DIMENSION on interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov)

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Main idea: use dynamic programming on a path-decomposition of G^4

- each bag has size $O(k^2)$.
- it suffices to separate vertices at distance 2 in G
- "transmission" lemma for separation constraints

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G^4 is an interval graph (with same left and right endpoint orders as G).

Each bag of a path-decomposition of G^4 is a clique in G^4.

Lemma: If $H \subset G$ has diameter D, then $|V(H)| = O(D \cdot k^2)$.

Note: in general graphs, $|V(H)| = O(D^k)$

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Definition (distance 2 resolving set): set that separates all pairs u, v with $d(u,v) \leq 2$.

Lemma: In an interval graph G, $R \subseteq V(G)$ is a resolving set if and only if R a distance 2 resolving set.

$\rightarrow$ Every pair to be separated will be present in some bag.

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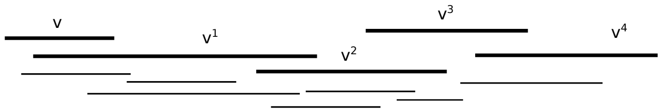
Definition (rightmost path $v, v^1, v^2, \ldots$ of v):



Lemma: vertex x separates u, v if and only if (for some/all i) x separates u^i, v^i.

→ Information about (non-)separation transmitted from bag to bag

FPT algorithm for METRIC DIMENSION on interval graphs

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Open problems

- Complexity of LD+MD for unit interval + bipartite permutation?
- Complexity of MD for bounded treewidth? (open for TW 2)
- Parameterized complexity of MD (parameter k)?
  $\rightarrow$ ~~permutation~~, ~~chordal~~, planar, line/claw-free...

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THANKS FOR YOUR ATTENTION